Chapter 6 - Day 2

Fermat's Theorem: let $f(x)$ be a continuous function on the interval $[a,b]$. If f has an extreme value at a point c strictly between a and b, and if f is differentiable at $x=c$, then $f'(c)=0$.

Corollary: Let $f(x)$ be a continuous function on the closed, bounded interval $[a,b]$. If f has an extreme value at $x=c$ in the interval, then either

- $c=a$ or $c=b$ (endpoints)
- $a<c<b$ and $f'(c)=0$ ($f'(c)=0$)
- $a<c<b$ and f is not differentiable at $x=c$, so that f' is not defined at $x=c$.

<u>Ex</u>: find the max and min values of $f(x) = x^3 + \frac{1}{2}x^2 - 2x + 6$ on the interval $[-2, 0]$.

find f' and set it equal to 0!

$f'(x) = 3x^2 + x - 2$

$3x^2 + x - 2 = 0$

$(3x-2)(x+1) = 0$

$x = \frac{2}{3}, -1$

2/3 not in interval - throw it out!

Since $f'(x)$ is a polynomial, its defined everywhere so we only need to check $x=-1$ and the endpoints.

$f(-1) = (-1)^3 + \frac{1}{2}(-1)^2 - 2(-1) + 6 = 7\frac{1}{2}$ ← global max

$f(-2) = (-2)^3 + \frac{1}{2}(-2)^2 - 2(-2) + 6 = 4$ ← global min

$f(0) = 0^3 + \frac{1}{2}(0^2) - 2(0) + 6 = 6$

max at $(-1, 7\frac{1}{2})$

min at $(-2, 4)$

Ex: Find the max and min values of $F(t) = \frac{3t+6}{t-2}$ on the interval $[3,7]$.

F(t) not defined at $t=2$, but 2 is not in the interval - don't worry about it!

$F'(t)$ quotient rule

$$F'(t) = \frac{(3)(t-2) - (3t+6)(1)}{(t-2)^2} = \frac{3t-6-3t-6}{(t-2)^2} = \frac{-12}{(t-2)^2}$$

$F'(t)$ does not exist at $t=2$, but outside interval ✓

$F'(t)$ never equals 0 ✓

check endpoints.

$$F(3) = \frac{3(3)+6}{3-2} = \frac{15}{1} = 15 \leftarrow \text{max}$$

$$F(7) = \frac{3(7)+6}{7-2} = \frac{27}{5} = 5.4 \leftarrow \text{min}$$

max at $(3,15)$ and a min at $(7, \frac{27}{5})$

Ex: Find the max and min of

$f(x) = x^{3/5}$ on $[-1, 32]$.

$f(x) = x^{3/5} = \sqrt[5]{x^3} = \left(\sqrt[5]{x}\right)^3$ defined everywhere

$f'(x) = \frac{3}{5} x^{3/5 - 1} = \frac{3}{5} x^{-2/5} = \frac{3}{5\sqrt[5]{x^2}}$

$f'(x)$ not defined at $x=0$

$f'(x)$ never equals 0 ✓

check endpoints and $x=0$

$f(0) = 0^{3/5} = 0$

$f(-1) = (-1)^{3/5} = \left((-1)^3\right)^{1/5} = -1$ ← min

$f(32) = (32)^{3/5} = \left(\sqrt[5]{32}\right)^3 = 2^3 = 8$ ← max

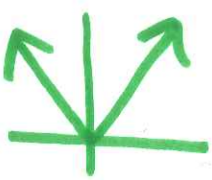
min at $(-1, -1)$

max at $(32, 8)$

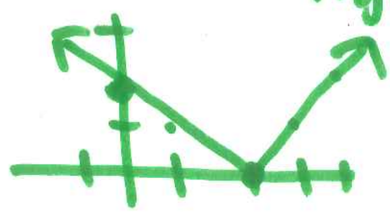
Ex: Consider $g(t) = |t-2| + 3$. Where is it's max and min on $[-5, 5]$?

We don't know the derivative of the absolute value function, so we need to rely on the graph.

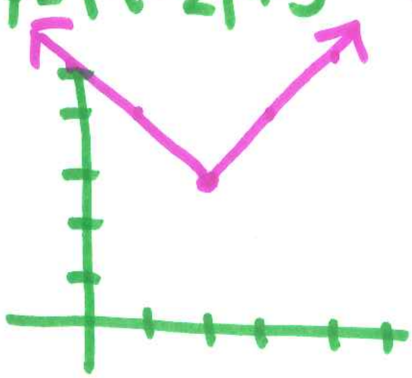
$y = |t|$



$y = |t-2|$ shifts right



$y = |t-2| + 3$ shifts up



min is at the vertex $(2, 3)$

max will be at an endpoint

$g(-5) = |-5-2| + 3 = 7 + 3 = 10 \leftarrow$ max

$g(5) = |5-2| + 3 = 3 + 3 = 6$

max occurs at $(-5, 10)$

Ex: find the max and min of $k(x)$ on the interval $[-3,4]$.

$$k(x) = \begin{cases} x^2+3x+1 & \text{if } x \le 1 \\ -4x+9 & \text{if } x > 1 \end{cases}$$

take the derivative and check the endpoints of each piece!

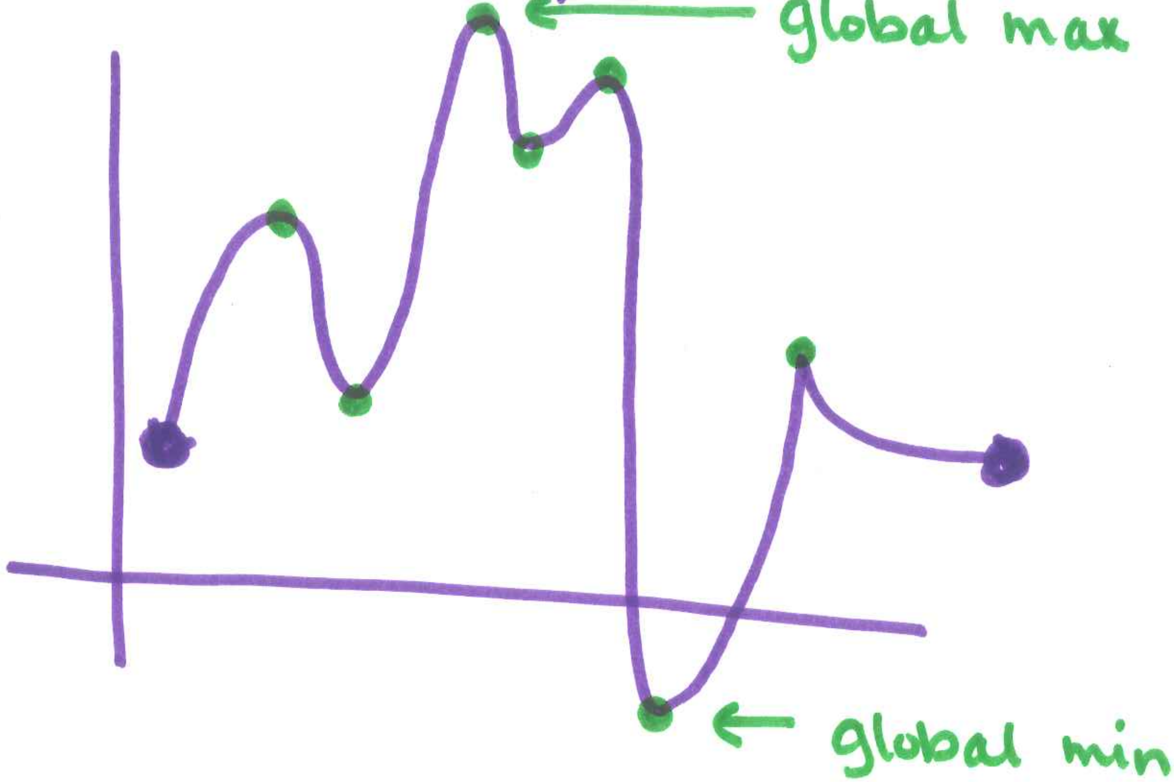
$$k'(x) = \begin{cases} 2x+3 & \text{if } x < 1 \\ -4 & \text{if } x > 1 \end{cases}$$

$k'(x) = 0$ if $2x+3=0$

$$2x = -3$$

$$x = -3/2$$

If we think about the graph of a landscape...



global max = highest hill

global min = lowest valley

but the other hills and valleys are important too!

defn: a function f has a local or relative maximum at a point $(c, f(c))$ if there is some interval about c such that $f(c) \geq f(x)$ for all x in the interval. A function f has a local or relative minimum at a point $(c, f(c))$ if there is some interval about c such that $f(c) \leq f(x)$ for all x in that interval.

Thm: if f has a local extreme value at $(c, f(c))$ and is differentiable at that point c, then $f'(c) = 0$.

Let f be a function. If f is defined at the point $x = c$ and either $f'(c) = 0$ or $f'(c)$ is undefined then the point c is called a Critical point of f.